

STABILISERS OF EIGENVECTORS OF FINITE REFLECTION GROUPS

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ABSTRACT. Let x be an eigenvector for an element of a finite irreducible reflection group W . Let W_x denote the subgroup of W which stabilises x . We provide an upper bound for the number of roots in the root system of W_x . This generalises a result of Kostant, who showed that every eigenvector with eigenvalue a primitive h^{th} root of unity is regular, where h is the Coxeter number of W . We also give a Lie-theoretic interpretation of our result in the study of semisimple conjugacy classes over Laurent series. In a forthcoming paper, we use this result to establish a geometric analogue of a conjecture of Gross and Reeder.

CONTENTS

1. Introduction	1
1.1. Recollections on reflection groups	1
1.2. Kostant's theorem on eigenvectors	2
1.3. Main theorem	2
1.4. Semisimple conjugacy classes	3
1.5. Acknowledgement	3
2. Parabolic subgroups	4
2.1. Recollections	4
2.2. Invariant quadratic polynomial	4
2.3. Eigenspaces	5
2.4. Parabolic eigenspaces	5
2.5. Non-regular elements	5
3. Proofs	6
3.1. Theorem 2 in type A_{n-1}	6
3.2. Theorem 2 in type B_n	7
3.3. Theorem 2 in type D_n	7
3.4. Theorem 2 in exceptional types	7
3.5. Theorem 3	8
References	8

1. INTRODUCTION

1.1. Recollections on reflection groups. Let V be a finite dimensional real vector space endowed with a positive definite symmetric bilinear form (\cdot, \cdot) . Let W be a finite subgroup of $\text{GL}(V)$

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generated by reflections. Thus, W is a finite (real) reflection group. By slightly modifying the standard definition of a root system¹, one can attach a root system Φ to W , cf. [Hum90]. Henceforth, we assume that Φ is irreducible.

The rank of W is, by definition, the dimension of the vector space V . We let n denote this integer. The ring of invariant polynomials $\mathbb{R}[V]^W = \text{Sym}(V^*)^W$ is generated by algebraically independent homogenous polynomials Q_1, \dots, Q_n . Let d_i denote the degree of Q_i . These integers are known as the *degrees* of W . We arrange these so that

$$d_1 \leq d_2 \leq \dots \leq d_n.$$

Note that Q_i 's are also algebraically independent generators for the invariant ring $\mathbb{C}[V_{\mathbb{C}}]^W$, where $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$.

The largest degree d_n is known as the *Coxeter number* of W and is denoted by h . A *Coxeter element* of W is an element which can be written as a product of all simple reflections, for some choice of a basis of simple roots. All Coxeter elements are conjugate; moreover, their order equals h . Furthermore, we have

$$|\Phi| = nh.$$

We refer the reader to [Bou68, V.6] or [Hum90] for these facts.

1.2. Kostant's theorem on eigenvectors. We are interested in the eigenvectors of elements of W . Let $w \in W$ and $\lambda \in \mathbb{C}$. An eigenvector for w with eigenvalue λ is a non-zero element $x \in V_{\mathbb{C}}$ such that

$$w \cdot x = \lambda x.$$

Recall that an element $x \in V_{\mathbb{C}}$ is said to be *regular* if its stabiliser in W is trivial; i.e., $W_x = \{1\}$.

Theorem 1 (Kostant). *Let $x \in V_{\mathbb{C}}$ be an eigenvector of an element of W with eigenvalue a primitive h^{th} root of unity. Then x is regular.*

Proof. The statement follows from three fundamental results of Kostant [Kos59, §9]. Since these results are proved in the setting of Weyl groups, we also give references for their generalisation to reflection groups:

- (i) Every eigenvalue λ must be a b^{th} root of unity, where b is a divisor of one of the degrees of W , cf. [Spr74, Theorem 3.4].
- (ii) If λ is a primitive h^{th} root of unity, then w is a Coxeter element of W , cf. [Kan01, §Theorem 32.C].
- (iii) Every eigenvector of a Coxeter element is *regular*; i.e., its stabiliser in W is trivial. The proof for finite reflection groups is implicit in, e.g., [Hum90, §3.19] or [Kan01, §29.6].

□

Our main theorem is a generalisation of Kostant's theorem, where h is replaced by an arbitrary positive integer.

1.3. Main theorem. For every $x \in V_{\mathbb{C}}$, let $W_x < W$ denote the stabiliser subgroup; i.e.,

$$W_x = \{w \in W \mid w \cdot x = x\}.$$

As we shall see in §2.1, W_x is a parabolic subgroup of W . Now let $\Phi_x \subseteq \Phi$ denote the root system of W_x . Set

$$N(x) := |\Phi| - |\Phi_x|.$$

¹For the standard definition see, for instance, [Bou68]. In the modified definition that is used in the context of reflection groups all roots have the same length; see [Hum90].

Equivalently, $N(x)$ is the number of roots in Φ whose pairing with x is non-zero; i.e.,

$$N(x) = |\{\alpha \in \Phi \mid (\alpha, x) \neq 0\}|.$$

Note that x is regular if and only if $N(x) = |\Phi| = nh$.

The following is our main result:

Theorem 2. *Let W be a finite irreducible reflection group of rank n . Let b a positive integer and let x be an eigenvector for an element of W with eigenvalue a primitive b^{th} root of unity. Then*

$$(1) \quad N(x) \geq bn.$$

Moreover, equality is achieved if and only if $b = h$.

We prove this theorem by using the classification of irreducible finite reflection groups; see §3. One can interpret this theorem as providing (yet another) characterisation of the Coxeter number.

1.4. Semisimple conjugacy classes. Let G be a simple simply connected algebraic group over \mathbb{C} . Let \mathfrak{g} denote its Lie algebra. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and let W denote the Weyl group.

Let \mathcal{K} be a field containing \mathbb{C} . The group $G(\mathcal{K})$ acts on $\mathfrak{g}(\mathcal{K})$ by the adjoint action. The orbits of this action are the conjugacy classes of $\mathfrak{g}(\mathcal{K})$. The set of conjugacy classes is denoted by $\mathfrak{g}(\mathcal{K})/G(\mathcal{K})$. Similarly, we have the conjugacy classes $\mathfrak{g}(\overline{\mathcal{K}})/G(\overline{\mathcal{K}})$. There is a canonical map

$$\mathfrak{g}(\mathcal{K})/G(\mathcal{K}) \rightarrow \mathfrak{g}(\overline{\mathcal{K}})/G(\overline{\mathcal{K}})$$

which sends a \mathcal{K} -conjugacy class to the unique $\overline{\mathcal{K}}$ -conjugacy class which contains it. It is natural to wonder what the image of this map is.

Restricting attention to *semisimple* conjugacy classes, the map in question can be written as

$$(2) \quad \mathfrak{g}(\mathcal{K})^{\text{ss}}/G(\mathcal{K}) \rightarrow \mathfrak{g}(\overline{\mathcal{K}})^{\text{ss}}/G(\overline{\mathcal{K}}) = \mathfrak{h}(\overline{\mathcal{K}})/W.$$

Here $\mathfrak{g}(-)^{\text{ss}} \subset \mathfrak{g}(-)$ is the subset of semisimple elements. The equality in the above line is the well-known bijection between semisimple conjugacy classes in $\mathfrak{g}(\overline{\mathcal{K}})$ and elements of $\mathfrak{h}(\overline{\mathcal{K}})/W$, cf. [CM93, Theorem 2.2.4]. Using Theorem 2, we can provide a constraint for the image of the above map, when $\mathcal{K} = \mathbb{C}((t))$ is the field of Laurent series.

Before stating our result, we recall that

$$\overline{\mathcal{K}} = \bigcup_{b \in \mathbb{Z}_{\geq 1}} \mathbb{C}((t^{\frac{1}{b}})).$$

Thus, every element of $\mathfrak{h}(\overline{\mathcal{K}})$ is in $\mathfrak{h}(\mathbb{C}((t^{\frac{1}{b}})))$ for some positive integer b .

Theorem 3. *Suppose $X \in \mathfrak{h}(\overline{\mathcal{K}})$ is conjugate to a point of $\mathfrak{g}(\mathcal{K})$.² Let $x \in \mathfrak{h} \setminus \{0\}$ be the leading term of X ; i.e.*

$$X = xt^{\frac{a}{b}} + \text{higher order terms}, \quad \gcd(a, b) = 1, \quad b > 0.$$

Then $N(x) \geq bn$. Moreover, equality is achieved if and only if b equals the Coxeter number of \mathfrak{g} .

We will prove this theorem by using Theorem 2 and a result of Springer on eigenspaces of reflection groups; see §3. In the forthcoming paper [KS16], we use Theorem 3 to prove a geometric analogue of a conjecture of Gross and Reeder [GR10]. We refer the reader to [RY14] and [GLRY12] for other intriguing connections between Lie theory and the Gross-Reeder story.

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²Equivalently the class of X in $\mathfrak{h}(\overline{\mathcal{K}})/W$ is in the image of the map defined in (2).

2. PARABOLIC SUBGROUPS

In this section, we gather some facts regarding parabolic subgroups of finite reflection groups. These facts are used in the proof of the main theorem in the *exceptional* types. The reader interested in the classical types (i.e., types A , B and D) will only need Lemma 4 from this section.

2.1. Recollections. We keep the notation introduced in the previous section. Let $\Delta \subset \Phi$ be a basis of simple roots. For every subset $I \subset \Delta$, let W_I denote the subgroup of W generated by I . Recall that a subgroup of W is said to be *parabolic* if it is of the form wW_Iw^{-1} for some $w \in W$.

Parabolic subgroups arise as stabilisers of the action of W on V ; that is, if $x \in V$ then W_x is a parabolic subgroup, cf. [Kan01, §5.2]. Similar result holds in the complex situation:

Lemma 4. *Let $x \in V_{\mathbb{C}}$. Then W_x is a parabolic subgroup of W .*

Proof. Write $x = a + bi$ where $a, b \in V$. The action of $w \in W$ is given by $w \cdot x = w \cdot a + (w \cdot b)i$. It follows that $W_x = W_a \cap W_b$. According to a result of Tits and Solomon (cf. [Sol76, Lemma 2] or [Qi07]), the intersection of two parabolic subgroups of W is again a parabolic subgroup. Thus, W_x is a parabolic subgroup. \square

Next, let V_I denote the \mathbb{R} -span of I in V . Let $\Phi_I = \Phi \cap V_I$. Then Φ_I is a root system in V_I with corresponding reflection group W_I , cf. [Hum90, §1.10].

Let U_I denote the orthogonal complement of V_I with respect to (\cdot, \cdot) . By definition, for every $\alpha \in I$ and $u \in U_I$ we have $(\alpha, u) = 0$. Thus, we have

$$(3) \quad s_{\alpha}(u) = u, \quad \alpha \in I, \quad u \in U_I,$$

where s_{α} is the reflection defined by the (simple) root α .

2.2. Invariant quadratic polynomial. Let W be a finite irreducible reflection group of rank $n > 2$. Let x be an element of $V_{\mathbb{C}}$. In this section, we show that if $Q(x) = 0$, where Q is the unique, up to scalar, invariant quadratic polynomial, then $\text{rank}(W_x) \leq n - 2$.

It is easy to write down a quadratic (homogenous) invariant polynomial. Namely, let e_1, \dots, e_n be an orthonormal basis of V with respect to (\cdot, \cdot) . Let $x \in V$ and write $x = \sum x_i e_i$. Then the polynomial $Q \in \mathbb{R}[V]$ defined by

$$Q(x) := \sum_{i=1}^n x_i^2$$

is invariant under the action of the orthogonal group. In particular, it is invariant under the action of $W < O(V)$. Moreover, every homogenous invariant polynomial of degree 2 is a scalar multiple of Q .

We can also regard $\{e_i\}$ as a \mathbb{C} -basis for $V_{\mathbb{C}}$. If $x \in V_{\mathbb{C}}$, we write $x = \sum x_i e_i$ where $x_i \in \mathbb{C}$. The polynomial $Q(x) = \sum x_i^2$ is also the unique, up to scalar, homogenous element of degree 2 in the invariant ring $\mathbb{C}[V_{\mathbb{C}}]^W$.

Lemma 5. *Let A be a subspace of V of dimension $n - 1$. Let x be a non-zero element in $V_{\mathbb{C}}$ satisfying $(x, y) = 0$ for all $y \in A_{\mathbb{C}}$. Then $Q(x) \neq 0$.*

Proof. Let $A^{\perp} \subset V$ denote the orthogonal complement of A with respect to (\cdot, \cdot) . Then A^{\perp} is a line in V . Without the loss of generality, suppose A^{\perp} is not in the hyperplane defined by setting the last coordinate equal to zero. Then there exists real numbers $r_1, \dots, r_{n-1}, r_n = 1$ such that

$$A^{\perp} = \{(a_1, \dots, a_n) \in V \mid a_i = r_i \cdot a_n\}$$

By assumption $x \in (A_{\mathbb{C}})^{\perp} = A^{\perp} \otimes_{\mathbb{R}} \mathbb{C}$; thus, we also have that $x_i = r_i x_n$ for $i = 1, 2, \dots, n$. Hence,

$$Q(x) = \sum_{i=1}^n x_i^2 = (1 + \sum_{i=1}^{n-1} r_i^2) x_n^2 \neq 0.$$

□

Corollary 6. *Let x be a non-zero element of $V_{\mathbb{C}}$ and suppose $Q(x) = 0$. Then $\text{rank}(W_x) \leq n - 2$*

Proof. By Lemma 4, W_x is a parabolic subgroup. Since x is non-zero, W_x is a proper subgroup. We may assume that $W_x = W_I$ for a set of simple roots I . This means that for all $\alpha \in I$, we have $s_{\alpha}(x) = x$ which, in turn, implies that $(x, \alpha) = 0$. It follows that $(x, y) = 0$ for all $y \in V_I$. By the previous lemma, $\dim(V_I) = \text{rank}(W_I) \leq n - 2$. □

2.3. Eigenspaces. Let ζ be a primitive b^{th} root of unity. For each $w \in W$, let $\Omega(w, \zeta) \subseteq V_{\mathbb{C}}$ denote the eigenspace of w with eigenvalue ζ ; i.e.

$$\Omega(w, \zeta) := \{x \in V_{\mathbb{C}} \mid w.x = \zeta.x\}.$$

Let

$$(4) \quad V(b) := \bigcup_{w \in W} \Omega(w, \zeta).$$

Thus, $V(b)$ is the set of all eigenvectors with eigenvalues a primitive b^{th} root of unity.

The following result is due to Springer; see [Spr74, §3].

Proposition 7. $V(b) = \{x \in V_{\mathbb{C}} \mid Q_i(x) = 0 \text{ for all } i \text{ such that } b \text{ does not divide } d_i\}.$

2.4. Parabolic eigenspaces. Let $P = W_I$ be a parabolic subgroup of W . Let $w \in P$. Recall that we have a natural subspace $V_I \subseteq V$ on which P acts. Let $\Omega_P(w, \zeta) \subseteq V_I \otimes_{\mathbb{R}} \mathbb{C}$ denote the eigenspace of w with eigenvalue ζ ; i.e.,

$$\Omega_P(w, \zeta) := \{x \in V_I \otimes_{\mathbb{R}} \mathbb{C} \mid w.x = \zeta.x\}.$$

We call this the *parabolic eigenspace* of $w \in W_I$. It is clear $\Omega_P(w, \zeta) \subseteq \Omega(w, \zeta)$. In fact, we have equality:

Lemma 8. *Suppose $\zeta \neq 1$. Then $\Omega_P(w, \zeta) = \Omega(w, \zeta)$.*

Proof. Let $x \in \Omega(w, \zeta)$. Recall that we have the orthogonal decomposition $V = V_I \oplus U_I$ and its complexification

$$V_{\mathbb{C}} = V_I \otimes_{\mathbb{R}} \mathbb{C} \oplus U_I \otimes_{\mathbb{R}} \mathbb{C}.$$

Let us write

$$x = v + u, \quad v \in V_I \otimes_{\mathbb{R}} \mathbb{C}, \quad u \in U_I \otimes_{\mathbb{R}} \mathbb{C}.$$

By (3), we have $w.u = u$. Thus,

$$w.v + u = w.v + w.u = w(v + u) = \zeta(v + u).$$

It follows that $u - \zeta.u \in (V_I \cap U_I) \otimes_{\mathbb{R}} \mathbb{C}$; hence, $u = 0$. It follows that $x \in V_I \otimes_{\mathbb{R}} \mathbb{C}$. □

2.5. Non-regular elements. Let b be a positive integer and ζ a primitive b^{th} root of unity. The following result is due to Springer, see [Spr74, Lemma 4.12].

Lemma 9. *Let $x \in V(b)$ be a non-regular non-zero element. Then $x \in \Omega(w, \zeta)$, where $w \in W$ is an element with eigenvalue one.*

Proposition 10. *Let b be an integer bigger than one and let $x \in V(b)$ be a non-zero non-regular element. Then, we have:*

- (i) *There exists a parabolic subgroup P and $w \in P$ such that $x \in \Omega_P(w, \zeta)$.*
- (ii) *b divides a degree of P .*

Proof. According to the previous lemma, $x \in \Omega(w, \zeta)$, where $w \in W$ is an element with eigenvalue one. Let $z \in V_{\mathbb{C}}$ denote the corresponding eigenvector. Let $P = W_z$ denote the isotropy group. By definition, $w \in P$. By Lemma 4, P is a parabolic subgroup and by Lemma 8, the parabolic eigenspace coincides with the usual one. Thus, $x \in \Omega_P(w, \zeta)$, as required.

Part (ii) follows from the fact that $\Omega_P(w, \zeta)$ is non-empty if and only if b divides a fundamental degree of P . \square

3. PROOFS

3.1. Theorem 2 in type A_{n-1} . Let n be an integer greater than one. Let V be the subspace of the n -dimensional affine space consisting of n -tuples (x_1, \dots, x_n) satisfying $\sum x_i = 0$. We use the elementary symmetric functions as invariant polynomials. Then $W \simeq S_n$.

3.1.1. Let $x \in V_{\mathbb{C}}$ be a non-zero element. According to Lemma 4, W_x is a proper parabolic subgroup. It is easy to check that the parabolic of W with the largest number of root is isomorphic to S_{n-1} . Thus, for all non-zero $x \in V_{\mathbb{C}}$, we have

$$|\Phi| - |\Phi_x| \geq n(n-1) - (n-1)(n-2) = 2(n-1) > (n-1).$$

Therefore, the theorem is evident for $b = 1$. Henceforth, we assume $1 < b < n$.

3.1.2. Suppose $x = (x_1, \dots, x_n) \in V(b)$. Then there exists $1 \leq k \leq \frac{n}{b}$ and complex numbers a_i such that the x_i 's are the roots of the polynomial

$$P(X) := X^n + a_1 X^{n-b} + \dots + a_k X^{n-bk}, \quad a_k \neq 0.$$

Now let $\alpha_1, \dots, \alpha_k$ denote the roots of the polynomial

$$Q(Y) := Y^k + a_1 Y^{k-1} + \dots + a_k.$$

Since $a_k \neq 0$ we have that $\alpha_i \neq 0$ for all i . Let ζ be a primitive b th root of unity. Then the roots of $P(X)$ equal

$$\zeta^i \alpha_j, \quad i \in \{1, 2, \dots, b\}, \quad j \in \{1, 2, \dots, k\}$$

together with $n - kb$ copies of 0.

3.1.3. Let us ignore the requirement $\sum x_i = 0$. Then the largest possible stabiliser for x is achieved if and only if $\zeta^{i_1} \alpha_1 = \zeta^{i_2} \alpha_2 = \dots = \zeta^{i_k} \alpha_k$ for some integers i_1, \dots, i_k . In this case,

$$W_x \simeq (S_k)^b \times S_{n-bk}.$$

Thus, for every non-zero $x \in V_{\mathbb{C}}$, we have

$$\begin{aligned} N(x) &= |\Phi| - |\Phi_x| \geq |\Phi_{S_n}| - |\Phi_{(S_k)^b \times S_{n-bk}}| \\ &= n(n-1) - [bk(k-1) + (n-kb)(n-kb-1)] \\ &= 2kbn - k^2b^2 - bk^2. \end{aligned}$$

3.1.4. We claim that the above quantity is bigger than $b(n-1)$. Indeed, if $k = 1$, then

$$N(x) \geq 2bn - b^2 - b > b(n-1),$$

where the last inequality follows from $b < n$. On the other hand, if $k > 1$, then

$$N(x) = 2kbn - k^2b^2 - bk^2 \geq b(2kn - kn - k^2) > b(n-1).$$

Here, the second to last inequality follows from the fact that $kb \leq n$. The last inequality is equivalent to

$$n(k-1) > (k-1)(k+2) \iff n > k+2$$

which is true because $1 < k \leq \frac{n}{2}$. This completes the proof in type A . \square

3.2. Theorem 2 in type B_n . As we saw above, the proof in type A has four steps. We sketch how these steps need to be modified in type B . Firstly, one checks that the parabolic with the largest number of roots is isomorphic to B_{n-1} . Thus, the theorem is evident for $b \leq 3$. Hence, we can assume $3 < b < 2n$. For the second step, one needs to consider the polynomial

$$X^{2n} + a_1 X^{2n-b} + \dots + a_k X^{2n-bk}, \quad a_k \neq 0.$$

For the third step, one checks that the largest possible stabiliser is

$$W_x \simeq (S_k)^b \times W_{B_{n-bk-1}}.$$

Thus,

$$N(x) = |\Phi| - |\Phi_x| \geq |\Phi_{B_n}| - |\Phi_{(A_{k-1})^b \times \Phi_{B_{n-bk-1}}}| \geq 5bkn - bk^2 - 2b^2k^2 - 2 + 2(n - bk).$$

Finally, it is easy to check that the above quantity is bigger than bn . \square

3.3. Theorem 2 in type D_n . Note $D_3 \simeq A_3$, so we may assume $n \geq 4$. It is easy to check that the parabolic subgroup of D_n with the largest number of roots is isomorphic to D_{n-1} . Thus, the theorem is evident for $b \leq 3$. Henceforth, we assume $4 < b < 2n$ and proceed as in type A . The polynomial we need to consider is

$$X^{2n} + a_1 X^{2n-b} + \dots + a_k X^{2n-bk}, \quad a_k \neq 0, \quad a_i \in \mathbb{C}.$$

One checks that the stabiliser with the maximum number of roots is

$$W_x \simeq (S_k)^b \times W_{D_{n-bk-1}}.$$

Therefore, we have

$$N(x) = |\Phi| - |\Phi_x| \geq |\Phi_{D_n}| - |\Phi_{(A_{k-1})^b \times \Phi_{D_{n-bk-1}}}| = 2n(n-1) - [bk(k-1) - 2(n-bk-1)(n-bk-2)].$$

Finally, one can show that the above quantity is bigger than bn . \square

3.4. Theorem 2 in exceptional types. We provide the complete proof in the case of E_6 . The proof in other exceptional types is similar, but easier. In more details, as we shall see, the proof in type E_6 has four steps. For types E_7 and E_8 , one only needs to imitate the first three step. For type F_4 one only needs the first two steps. Finally, for types G , H and I_2 , one only needs to imitate the first step.

3.4.1. Recall that E_6 has 72 roots and its fundamental degrees are 2, 5, 6, 8, 9, 12. It is easy to check that the proper parabolic of E_6 with the largest number of roots is D_5 with 40 roots. Thus, for all non-zero $x \in V_{\mathbb{C}}$, we have

$$N(x) \geq 72 - 40 = 32 > 5 \times 6.$$

Hence, the theorem is evident for $b \leq 5$.

3.4.2. Henceforth, we assume $b > 5$. Let Q denote the unique, up to scalar, homogenous quadratic invariant polynomial; see §2.2. If x is an eigenvector with eigenvalue a b^{th} root of unity, then $Q(x) = 0$. By Corollary 6, the stabiliser W_x must be a parabolic subgroup of rank ≤ 4 . The maximum number of roots in a parabolic subgroup of E_6 of rank ≤ 4 is 24 (for D_4). Thus,

$$(5) \quad N(x) \geq 72 - 24 = 48 > 6 \times 6.$$

Hence, our result also holds for $b = 6$.

3.4.3. The remaining cases are $b = 8$ and $b = 9$. Let $x \in V(9)$ be a non-zero element. According to Proposition 10, if x is not regular, then there must exists a proper parabolic subgroup of W with a fundamental degree divisible by 9. But there is no such parabolic subgroup of E_6 . Thus, x must be regular, and so the theorem is immediate.

3.4.4. It remains to treat the case $b = 8$. Suppose x is a non-regular non-zero element of $V(8)$. According to Proposition 10, there exists a parabolic subgroup $P = W_I < W$ and $w \in P$ such that

$$x \in \Omega(w, \zeta) = \Omega_P(w, \zeta) \subseteq V_I,$$

where ζ is a primitive 8^{th} root of unity. Moreover the parabolic P must have a fundamental degree divisible by 8. The only possibility is $P \simeq D_5$. But in this case, 8 is the *highest* fundamental degree of P and so, by Kostant's theorem (Theorem 1), $x \in V_I$ is *regular* for the reflection action P on V_I . In other words,

$$(\alpha, x) \neq 0, \quad \forall \alpha \in I.$$

Since $|I| = 5$, it follows that (α, x) is zero for at most one simple root of W . Hence, either x is regular or $W_x \simeq A_1$. In both cases, we have $N(x) > 6 \times 8$. \square

3.5. **Theorem 3.** In view of Theorem 2, it is sufficient to show that $x \in V(b)$; i.e., x is an eigenvector for some element of W with eigenvalue a b^{th} root of unity.

Let $Q \in \mathbb{C}[\mathfrak{g}]^G$ be an invariant homogenous polynomial of degree d . Note that Q is also an invariant polynomial on $\mathfrak{g}(\mathcal{K})$ and $\mathfrak{g}(\overline{\mathcal{K}})$. Let $Y \in G(\overline{\mathcal{K}}).X \cap \mathfrak{g}(\mathcal{K})$. Since X and Y are conjugate, we have $Q(X) = Q(Y)$. On the other hand, Y is a \mathcal{K} -rational point; thus, $Q(X) \in \mathcal{K}$.

Now observe that

$$Q(X) = Q(xt^{\frac{a}{b}}) + \text{higher order terms} = t^{\frac{da}{b}} Q(x) + \text{higher order terms}.$$

By assumption $\gcd(a, b) = 1$; thus, for the leading term of the above expression to be in $\mathcal{K} = \mathbb{C}((t))$, we must have

$$(6) \quad Q(x) = 0, \quad \text{whenever } b \text{ does not divide } d.$$

By a theorem of Chevalley, the restriction map provides an isomorphism $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$. Thus, (6) is also satisfied for all homogenous polynomials $Q \in \mathbb{C}[\mathfrak{h}]^W$ of degree d . By Proposition 7, $x \in V(b)$. \square

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